

Revisiting f_B and $\bar{m}_b(\bar{m}_b)$ from HQET spectral sum rules

Stephan Narison^a

^aLaboratoire Particules et Univers de Montpellier, CNRS-IN2P3, Case 070, Place Eugène Bataillon, 34095 - Montpellier, France.

Abstract

Using recent values of the QCD (non-) perturbative parameters given in Table 1, we reconsider the extraction of f_B and the on-shell mass M_b from HQET Laplace spectral sum rules known to N2LO PT series and including dimension 7 condensates in the OPE. We especially study the convergence of the PT series, the effects on “different spectral sum rules data” of the continuum threshold and subtraction point varied in a larger range than in the existing literature and include in the error an estimate of the N3LO PT series based on a geometric growth of the PT series. We obtain the Renormalization Group Invariant (RGI) *universal coupling* : $\hat{f}_B^\infty = 0.407(28) \text{ GeV}^{3/2}$ in the static limit $M_b \rightarrow \infty$ and the physical decay constant including $1/M_b$ corrections: $f_B^{\text{hqet}} = 197(13) \text{ MeV}$. Using the ratio of sum rules, we obtain the on-shell b quark mass $M_b = 4871(30) \text{ MeV}$ to order α_s^2 from which we deduce the running mass $\bar{m}_b(\bar{m}_b) = 4234(56) \text{ MeV}$. The previous results are in good agreement with the ones from QCD spectral sum rules (QSSR) in full QCD to the same order [1]: $f_B^{qcd} = 206(7) \text{ MeV}$ and $\bar{m}_b(\bar{m}_b)^{qcd} = 4236(69) \text{ MeV}$ from the same channel. Using in addition the value $\bar{m}_b(\bar{m}_b) = 4177(11) \text{ MeV}$ from the Υ sum rules to order α_s^3 [2], we deduce the *final averaged estimate from QSSR*: $f_B^{qssr} = 204(6) \text{ MeV}$ and $\bar{m}_b(\bar{m}_b)^{qssr} = 4181(11) \text{ MeV}$.

Keywords: QCD spectral sum rules, meson decay constants, heavy quark masses, heavy quark effective theory.

1. Introduction

The (pseudo)scalar meson decay constants f_P are of prime interests for understanding the realizations of chiral symmetry in QCD. In addition to the well-known values of $f_\pi = (130.4(2) \text{ MeV})$ and $f_K = 156.1(9) \text{ MeV}$ [3] which control the light flavour chiral symmetries, it is also desirable to extract the ones of the heavy-light charm and bottom quark systems with high-accuracy. These decay constants are normalized through the matrix element:

$$\langle 0 | J_{qQ}^P(x) | P \rangle = f_P M_P^2, \quad (1)$$

where:

$$J_{qQ}^P(x) \equiv (m_q + M_Q) \bar{q}(i\gamma_5) Q, \quad (2)$$

is the local heavy-light pseudoscalar current; $q \equiv d, s$; $Q \equiv c, b$; $P \equiv D_{(s)}, B_{(s)}$ and where f_P is related to the leptonic width:

$$\Gamma(P^+ \rightarrow l^+ \nu_l) = \frac{G_F^2}{8\pi} |V_{Qq}|^2 f_P^2 m_l^2 M_P \left(1 - \frac{m_l^2}{M_P^2}\right)^2, \quad (3)$$

where m_l is the lepton mass and $|V_{Qq}|$ the CKM mixing angle. In a recent analysis [1], we have revised the extraction of these heavy-light decay constants in full QCD [1] using QCD spectral sum rules [4–9]. Here, we pursue the analysis by revisiting the determination of f_B from HQET spectral sum rules. In so doing, we shall explicitly analyze the influence on the results of the subtraction point μ , of the continuum threshold t_c . We

shall also use (besides recent determinations of the QCD input parameters) the new precise value of m_b from the Υ sum rules [2]. In addition, we shall re-extract the meson-quark mass difference using HQET sum rules from which we shall deduce the running b -quark mass.

2. HQET preliminaries

HQET spectral sum rules have been initially used by Shuryak [11] using a non-relativistic ¹ version of the NSVZ [12] sum rules in the large M_b limit. Shuryak's sum rule has been applied later on in HQET [13] by several authors [14–20]. The most important input in the analysis of HQET sum rule is local heavy-light quark axial-vector current of the full QCD theory which can be expressed as an OPE of the HQET operators \tilde{O}_n in the inverse of the heavy quark mass:

$$J_A^\mu(x, M_b) = C_b \left(\frac{M_b}{\mu}, \alpha_s(\mu) \right) \tilde{J}_A^\mu(x, M_b = \infty) + \sum_{n=1} C_n \left(\frac{M_b}{\mu}, \alpha_s(\mu) \right) \frac{\tilde{O}_n(M_b = \infty, \mu)}{M_b^n}, \quad (4)$$

where : $\tilde{J}_A^\mu \equiv \bar{q}\gamma^\mu\gamma^5 h_\nu$ is the quark current in HQET built from a light antiquark field \bar{u} and a properly normalized heavy quark field h_ν [13], $C_{b,n}$ are Wilson coefficients and M_b is the on-shell b -quark mass. Using a non-covariant normalization of hadronic

Email address: snarison@yahoo.fr (Stephan Narison)

¹Some earlier attempt to use a non-relativistic approach for estimating f_D can e.g. be found in [10].

states [21], one can define an universal coupling in the static limit:

$$\langle 0 | \tilde{J}_A^\mu | P(v) \rangle = \frac{i}{\sqrt{2}} \tilde{f}_{stat} v^\mu. \quad (5)$$

The coefficient function $C_b(M_b/\mu, \alpha_s(\mu))$ is obtained by requiring that HQET reproduces the full QCD theory at $\mu = M_Q$. It has been obtained to order α_s in [17] and to order α_s^2 in [22]. It reads in the \overline{MS} -scheme:

$$C_b(M_b) = 1 - \frac{2}{3} a_s + a_s^2 \left[-\frac{1871}{1729} - \frac{17\pi^2}{72} - \frac{\pi^2}{18} \ln 2 - \frac{11}{36} \zeta(3) + n_l \left(\frac{47}{288} + \frac{\pi^2}{36} \right) \right], \quad (6)$$

with: $a_s \equiv \alpha_s/\pi$. The HQET current \tilde{J}_A^μ acquires anomalous dimension, which reads to $O(\alpha_s^2)$ [13, 23, 24] (in our normalizations):

$$\begin{aligned} \gamma &\equiv \gamma_1 a_s + \gamma_2 a_s^2 + \dots, \\ \gamma_1 &= 1, \quad \gamma_2 = \frac{127}{72} + \frac{7\pi^2}{54} - \frac{5}{36} n_l. \end{aligned} \quad (7)$$

Therefore, the universal coupling scales as:

$$\tilde{f}_{stat}(\mu) = \frac{R_b(M_b)}{R_b(\mu)} \tilde{f}_{stat}(M_b), \quad (8)$$

with:

$$R_b(\mu) = (\alpha_s(\mu))^{-\gamma_1/\beta_1} \left[1 - \left(\frac{\gamma_2}{\beta_1} - \gamma_1 \frac{\beta_2}{\beta_1^2} \right) a_s(\mu) \right], \quad (9)$$

where the two first coefficients of the β functions are:

$$\beta_1 = -\frac{1}{2} \left(11 - \frac{2}{3} n_l \right), \quad \beta_2 = -\frac{1}{4} \left(51 - \frac{19}{3} n_l \right). \quad (10)$$

The *universal* coupling is connected to the physical decay constant as:

$$f_B \sqrt{M_B} = C_b(M_b) \tilde{f}_{stat}(M_b) + O(1/M_b). \quad (11)$$

It is also convenient to introduce the *universal* Renormalization Group Invariant (RGI) current and the associated coupling:

$$\hat{J}^\mu = R_b(\mu) \tilde{J}^\mu(\mu), \quad \hat{f}_B = R_b(\mu) \tilde{f}_{stat}(\mu), \quad (12)$$

which we shall estimate in the following.

3. HQET spectral sum rules for \hat{f}_B in the static limit

We shall be concerned with the universal 2-point-function ²:

$$\hat{\Pi}(q^2 \equiv -Q^2) = i \int d^4x e^{iqx} \langle 0 | \hat{J}(x) \hat{J}^\dagger(0) | 0 \rangle \quad (13)$$

²In HQET Lorentz structure is unimportant and it is only the parity which counts such that we shall omit it in the following.

for determining the coupling \hat{f}_B using QCD spectral sum rules (QSSR). Like in the case of the full theory, we can use either the Laplace (LSR) [4, 25, 26]:

$$\begin{aligned} \mathcal{L}(\tau, \mu) &\equiv \lim_{Q^2, n \rightarrow \infty} \frac{(-Q^2)^n}{(n-1)!} \frac{\partial^n \hat{\Pi}}{(\partial Q^2)^n} \\ &\quad n/Q^2 \equiv \tau \\ &= \tau \int_{M_b^2}^{\infty} dt e^{-t\tau} \frac{1}{\pi} \text{Im} \hat{\Pi}(t, \mu), \end{aligned} \quad (14)$$

or the $Q^2 = 0$ Moments sum rules (MSR) [4]:

$$\mathcal{M}^{(n)}(\mu) \equiv \frac{(-1)^n}{n!} \frac{\partial^n \hat{\Pi}}{(\partial Q^2)^n} \Big|_{Q^2=0} = \int_{M_b^2}^{t_c} \frac{dt}{t^{n+2}} \frac{1}{\pi} \text{Im} \hat{\Pi}(t, \mu). \quad (15)$$

However, the use of the $Q^2 = 0$ -moment sum rules is rather delicate as they do not have a proper infinite heavy quark mass limit. We shall not consider these sum rules here ³. For the present analysis, it is convenient to introduce respectively the soft scale, the meson-quark mass-difference and the HQET Laplace sum rule variable:

$$\omega = \frac{(q^2 - M_b^2)}{M_b}, \quad \Delta M = \frac{(M_B^2 - M_b^2)}{M_b}, \quad \tau_H = \tau M_b, \quad (16)$$

where M_b is the on-shell quark mass and τ is the usual LSR variable used in the full QCD theory and has the dimension of GeV^{-2} . As usual, we parametrize the spectral function using the Minimal Duality Ansatz (MDA):

$$\frac{1}{\pi} \text{Im} \hat{\Pi}(t) \simeq \hat{f}_B^2 \delta(t - M_B^2) + \text{“QCD cont.”} \theta(t - t_c), \quad (17)$$

where the accuracy for the sum rule approaches has been explicitly tested from heavy quarkonia systems in [1]. The perturbative (PT) expression of the spectral function has been evaluated to order $O(\alpha_s)$ (NLO) in [17] and to $O(\alpha_s^2)$ (N2LO) in [27]. It reads:

$$\begin{aligned} \text{Im} \hat{\Pi}_{PT}(\omega) &= \frac{3\omega^2}{8\pi} \left\{ 1 + a_s(\mu) \left(\frac{17}{3} + \frac{4\pi^2}{9} + L_\omega \right) \right. \\ &\quad + a_s^2(\mu) \left[99(15) + \left(\frac{1657}{72} + \frac{97\pi^2}{54} \right) L_\omega \right. \\ &\quad + \frac{15}{8} L_\omega^2 + n_l \left[-3.6(4) \right. \\ &\quad \left. \left. \left. - \left(\frac{13}{12} + \frac{2\pi^2}{27} \right) L_\omega - \frac{L_\omega^2}{12} \right] \right] \right\} \end{aligned} \quad (18)$$

with: $L_\omega \equiv \ln(\mu^2/\omega^2)$. We estimate the $O(\alpha_s^3)$ (N3LO) by assuming the geometric growth of the PT series [28] as a dual to the effect of a $1/q^2$ term [29, 30] which parametrizes the UV renormalon contributions.

NP corrections up to dimension 7 condensates has been obtained in [16, 31]. Including all the previous corrections the sum rule read for $M_b \rightarrow \infty$:

$$\hat{f}_B^2 = e^{\tau_H \Delta M} R_b^2(\mu) \left[\frac{1}{\pi} \int_0^{\omega_c} d\omega e^{-\omega \tau_H} \text{Im} \hat{\Pi}_{PT}(\omega) + NP \right], \quad (19)$$

³Some attempts to use $Q^2 = 0$ -moment sum rules can be found in [20].

where: $\omega_c = (t_c - M_b^2)/M_b$ and:

$$NP(\mu) = -\langle \bar{u}u \rangle(\mu) \left[1 + 2a_s(\mu) - \frac{M_0^2}{4} \tau_H^2 + \frac{\pi \langle \alpha_s G^2 \rangle}{18} \tau_H^4 \right] + \left(\pi \rho \langle \bar{u}u \rangle^2 - \frac{3g \langle G^3 \rangle}{356\pi^2} \right) \frac{2}{81} \tau_H^3. \quad (20)$$

4. The QCD input parameters

The QCD parameters which shall appear in the following analysis will be the on-shell bottom quark mass M_b (we shall neglect the light quark masses $q \equiv u$ here and in the following), the light quark condensate $\langle \bar{q}q \rangle$, the gluon condensates $\langle g^2 G^2 \rangle \equiv \langle g^2 G_{\mu\nu}^a G_{\mu\nu}^{a\prime} \rangle$ and $\langle g^3 G^3 \rangle \equiv \langle g^3 f_{abc} G_{\mu\nu}^a G_{\nu\rho}^b G_{\rho\mu}^c \rangle$, the mixed condensate $\langle \bar{q}g\sigma Gq \rangle \equiv \langle \bar{q}g\sigma^{\mu\nu}(\lambda_a/2)G_{\mu\nu}^a q \rangle = M_0^2 \langle \bar{q}q \rangle$ and the four-quark condensate $\rho \langle \bar{q}q \rangle^2$, where $\rho \simeq 2$ indicates the deviation from the four-quark vacuum saturation. Their values are given in Table 1. We shall work with the running light quark parameters known to order α_s^3 [6, 7, 32]:

$$\begin{aligned} \bar{m}_{q,Q}(\mu) &= \hat{m}_{q,Q}(-\beta_1 a_s)^{-2/\beta_1} \times C(a_s) \\ \langle \bar{q}q \rangle(\mu) &= -\hat{\mu}_q^3 (-\beta_1 a_s)^{2/\beta_1} / C(a_s) \\ \langle \bar{q}g\sigma Gq \rangle(\mu) &= -M_0^2 \hat{\mu}_q^3 (-\beta_1 a_s)^{1/3\beta_1} / C(a_s), \end{aligned} \quad (21)$$

$\hat{m}_{q,Q}$ is the RGI quark mass, $\hat{\mu}_q$ is spontaneous RGI light quark condensate [33]. The QCD correction factor $C(a_s)$ in the previous expressions is numerically:

$$\begin{aligned} C(a_s) &= 1 + 0.8951a_s + 1.3715a_s^2 + \dots : n_f = 3, \\ &= 1 + 1.1755a_s + 1.5008a_s^2 + \dots : n_f = 5, \end{aligned} \quad (22)$$

which shows a good convergence. We shall use:

$$\alpha_s(M_\tau) = 0.325(8) \implies \alpha_s(M_Z) = 0.1192(10) \quad (23)$$

from τ -decays [34, 35], which agree perfectly with the world average 2012 [36, 37]:

$$\alpha_s(M_Z) = 0.1184(7). \quad (24)$$

The value of the running $\langle \bar{q}q \rangle$ condensate is deduced from the value of $(\bar{m}_u + \bar{m}_d)(2) = (7.9 \pm 0.6)$ MeV obtained in [38, 39] and the well-known GMOR relation: $(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle = -m_\pi^2 f_\pi^2$. We shall use the value of the RGI spontaneous mass to order α_s for consistency with the known α_s correction in the OPE:

$$\hat{\mu}_u = 251(6) \text{ MeV}. \quad (25)$$

The values of the running \overline{MS} mass $\bar{m}_b(m_b)$ recently obtained in Ref. [2] from bottomium sum rules, will also be used⁴. Using the relation between the running $\bar{m}_b(m_b) = 4177(11)$ MeV from the Υ -systems [2] and the on-shell (pole) M_b masses (see e.g. [6, 7, 27]), one can deduce to order α_s^2 :

$$M_b = 4804(50)_{\alpha_s} \rightarrow \alpha_s(M_b) = 0.2326(22), \quad (26)$$

where the error is mainly due to the one of α_s . This large error has to be contrasted with the precise value of the running

mass, and can be an obstacle for a precise determination of \hat{f}_B from HQET at a given μ . One can see in Section 8 that a direct extraction of the on-shell mass from the HQET at the same α_s^2 order leads to about the same value and error which is an (a posteriori) self-consistency check of the value and error used in Eq. (26) for the analysis. We are aware that the inclusion of the known α_s^3 -correction and an estimate of the PT higher order terms using a geometric growth of the PT coefficients à la Ref. [28, 29] increase the value of M_b by about (100 ~ 200) MeV, which could be considered if one works to higher order in α_s^n ($n \geq 3$).

Table 1: QCD input parameters.

Parameters	Values	Ref.
$\alpha_s(M_\tau)$	0.325(8)	[34–36]
$\bar{m}_b(\bar{m}_b)$	4177(11) MeV	average [2]
$\frac{1}{2}(\bar{u}u + \bar{d}d)^{1/3}(2)$	$-(275.7 \pm 6.6)$ MeV	[6, 38]
M_0^2	$(0.8 \pm 0.2) \text{ GeV}^2$	[42–44]
$\langle \alpha_s G^2 \rangle$	$(7 \pm 1) \times 10^{-2} \text{ GeV}^4$	[2, 26, 34, 45–51]
$\langle g^3 G^3 \rangle$	$(8.2 \pm 1.0) \text{ GeV}^2 \times \langle \alpha_s G^2 \rangle$	[2]
$\rho \langle \bar{q}q \rangle^2$	$(4.5 \pm 0.3) \times 10^{-4} \text{ GeV}^6$	[34, 42, 45]

5. The LSR determination of the RGI \hat{f}_B in the static limit

Analysis of the convergence of the PT series

We study the effect of the truncation of the PT series on the value of \hat{f}_B from Eqs. (19) and (20). For a given value of $\mu = M_b$ and $\omega_c = 3$ GeV, we show the result of the analysis in Fig. 1. At the minimum (optimal) value, \hat{f}_B moves from 0.365 (LO+NLO) to 0.414 (+N2LO) to 0.454 (+N3LO) $\text{GeV}^{3/2}$, i.e. a change of about 13% from LO+NLO to N2LO and of about 8.8% from N2LO to N3LO, which indicates a slow convergence of the PT series. We consider the N3LO contribution as a systematic error from the truncation of the PT series.

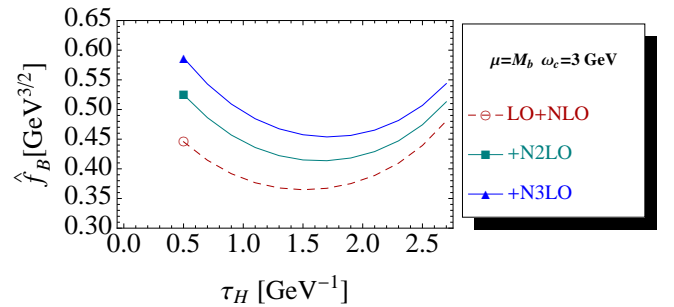


Figure 1: a) τ_H -behaviour of \hat{f}_B for $\mu = M_b$ and $\omega_c = 3$ GeV for different truncations of the PT series.

Analysis of the τ_H and ω_c stabilities

We show in Fig. 2 the τ_H -behaviour of the result for a given value of μ and for different values of ω_c where the PT series is known to N2LO. A τ_H -stability is obtained for $\omega_c \geq 2$ GeV,

⁴These values agree and improve previous sum rules results [4–7, 40, 41].

while ω_c -stability is obtained for $\omega_c \geq 4$ GeV. We consider as optimal and conservative values the ones obtained in the previous range of ω_c values:

$$\hat{f}_B(M_b) = (0.407 - 0.413) \text{ GeV}^{3/2}. \quad (27)$$

One should notice that contrary to some results obtained in the literature, we allow a larger range of ω_c -values from 2 GeV (beginning of τ_H -stability) to 4 GeV (beginning of ω_c -stability) in order to deduce a more conservative estimate.

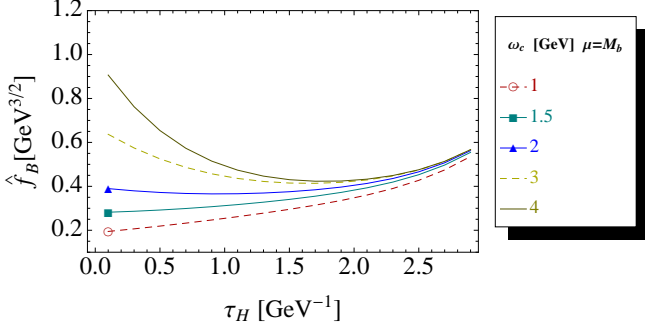


Figure 2: **a)** τ_H -behaviour of \hat{f}_B for $\mu = M_b$ and $\omega_c = 3$ GeV for different truncations of the PT series .

Summary of the results for \hat{f}_B and error calculations

At a given value of μ we estimate the errors induced by the QCD parameters compiled in Table 1. We summarize the results of the analysis in Table 2. We show in Fig 3 the “sum

Table 2: Central values and corresponding errors for \hat{f}_B in units of $10^3 \times \text{GeV}^{3/2}$ from the LSR at different values of the subtraction point μ in units of GeV for $M_b = 4804(50)$ MeV. The +(resp. -) sign means that the values of \hat{f}_B increase (resp. decrease) when the input increases (resp. decreases). The total error comes from a quadratic sum.

μ	\hat{f}_B	t_c	α_s	α_s^3	M_b	$\langle \bar{u}u \rangle$	$\langle G^2 \rangle$	M_0^2	Total
1	502	+20	+13	+111	-51	+5	+2	-22	125
2	437	+26	+6	+60	-39	+5	+2	-11	77
3	414	+27	+5	+50	-33	+4	+1	-8	67
4	401	+28	+4	+44	-31	+5	+1	-6	61
M_b	395	+28	+2	+40	-32	+4	+1	-7	60
5	392	+29	+4	+41	-29	+5	+1	-4	59
6	386	+29	+3	+38	-29	+5	+1	-4	56

rules data points” at different values of the subtraction point μ . The error is large at small μ due to the bad behaviour of the PT series at low scale which confirms the scepticism of the authors of Ref. [24] on the reliable extraction of \hat{f}_B at a such low scale. However, as we have shown in previous section, the convergence of the PT series improves obviously at larger scale which enables to extract \hat{f}_B with a reasonable accuracy. Fitting the previous data by an horizontal line or taking their average, we deduce the final value of the RGI universal coupling:

$$\hat{f}_B = 0.405(25) \text{ GeV}^{3/2}, \quad (28)$$

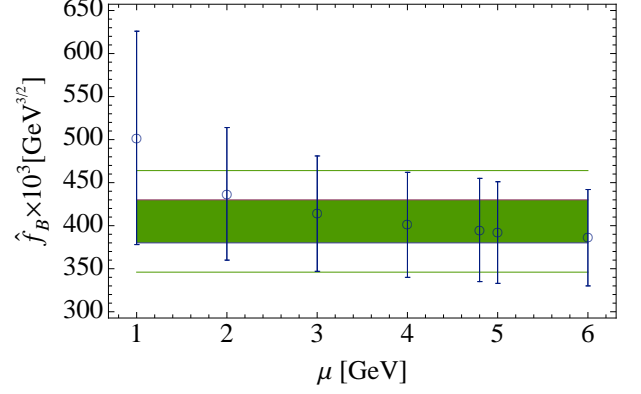


Figure 3: \hat{f}_B at N2LO for different values of the subtraction point μ from Table 2. The green coloured region corresponds to the one where the error comes from a weighted average, while the non-coloured one delimited by the two green horizontal lines is the one where the error coming from the best determination has been taken.

from which we can deduce the value of the static coupling evaluated e.g. at $M_B = 5.28$ GeV:

$$\tilde{f}_{stat}(M_B) = 0.587(2)_{\alpha_s(37)} \hat{f}_B \text{ GeV}^{3/2}. \quad (29)$$

The corresponding decay constant from Eq. (11):

$$f_B^\infty = 228(1)_{\alpha_s(14)} \hat{f}_B \text{ MeV}, \quad (30)$$

which is quite large compared to the value of $f_B = 206(7)$ MeV obtained from the full QCD theory [1] suggesting some large $1/M_b$ -corrections which we shall analyze in the next section. We consider the previous results in Eqs. (28) to (30) as improvements of previous results in the literature [15–20].

6. $1/M_b$ corrections and value of the decay constant f_B

Taking into account the mass-difference between the meson M_B and the on-shell quark mass M_b , the relation in Eq. (11) expressed in terms of the RGI coupling in Eq. (12) becomes:

$$f_B^2 = \left(\frac{M_b}{M_B} \right)^3 \left[\frac{C_b(M_b)}{R_b(M_b)} \frac{\hat{f}_B^2}{M_B} + \delta f_B^2 \right]. \quad (31)$$

LSR expression of $1/M_b$ correction δf_B^2

The $1/M_b$ corrections δf_B^2 to the HQET two-point correlator can be obtained by subtracting its expression in the full theory with the one of HQET in the limit $M_b \rightarrow \infty$. The $1/M_b$ correction to the physical decay constant f_B reads (see e.g. [20]):

$$\delta f_B^2 = \frac{e^{\tau_H \Delta M}}{M_B} \left[\frac{1}{\pi} \int_0^{\omega_c} d\omega e^{-\omega \tau_H} \text{Im} \delta \Pi_{PT}(\omega) + \delta_{NP} \right], \quad (32)$$

where:

$$\text{Im} \delta \Pi_{PT} = \text{Im} \Pi_{PT} - C_b(M_b) \frac{R_b(\mu)}{R_b(M_b)} \text{Im} \hat{\Pi}_{PT}. \quad (33)$$

Up to order α_s , it reads:

$$\text{Im}\delta\Pi_{PT}(x) = -\frac{3}{8\pi} \frac{1}{M_b^2} \frac{x^3}{1+x} \left\{ 1 + a_s \left[\frac{1}{2} \left(\frac{13}{4} + \frac{\pi^2}{3} - \frac{3}{2} \ln x \right) - \frac{F(x)}{x} \right] \right\} \quad (34)$$

where : $x \equiv \omega/M_b$ and:

$$F(x) = 2\text{Li}_2(-x) + \ln(x) \ln(1+x) - \frac{x}{1+x} \ln(x) + \frac{1+x}{x} \ln(1+x) - 1. \quad (35)$$

The order N2LO α_s^2 PT correction to the spectral function can be numerically obtained by subtracting the complete expression in full QCD obtained in [27] with the HQET asymptotic result in Eq. (18) and by using the relation in Eq. (32). In the same way, we estimate the N3LO PT corrections assuming a geometric growth of the PT series both in full QCD and HQET theories.

The NP corrections read up to d=5 condensates [17, 20]:

$$\delta_{NP}(\mu) = 2a_s(\mu) \langle \bar{u}u \rangle(\mu) \int_0^\infty \frac{d\omega}{M_b} \frac{e^{-\omega\tau_H}}{1+\omega/M_b} + \frac{\langle \alpha_s G^2 \rangle}{12\pi M_b} - \left(\frac{\tau}{2M_b} \right) \langle \bar{q}g\sigma Gq \rangle(\mu). \quad (36)$$

Analysis of the convergence of the PT series

Like in the case of \hat{f}_B , we study the convergence of the PT series. We notice that the α_s^2 and α_s^3 corrections are very small for $\tau_H \leq 1$ GeV, which can then be neglected. The analysis is shown in Fig. 4.

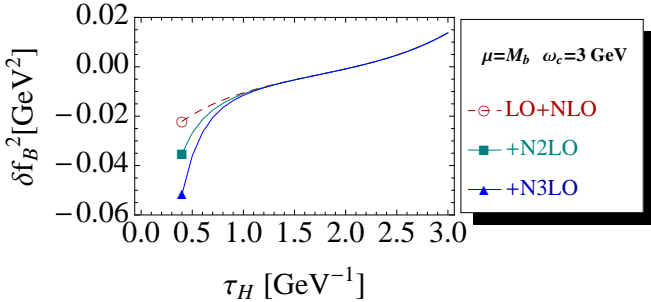


Figure 4: τ_H -behaviour of δf_B^2 for $\mu = M_b$ and $\omega_c = 3$ GeV for different truncations of the PT series where the contributions of condensates up to d=5 have been included.

Analysis of the τ_H and ω_c stabilities

We show in Fig. 5 the τ_H -behaviour of δf_B^2 for different values of ω_c and including the $d = 5$ condensates. We study the effects of the $d = 5$ condensates on the τ_H -stability for given two extremal values of ω_c (beginning of τ_H and of ω_c -stabilities). The analysis is shown in Fig. 6 from which we consider as optimal results the ones corresponding to the range:

$$\tau_H \simeq (1.8 \sim 2.2) \text{ GeV}^{-2}, \quad (37)$$

considering the fact that the inflexion point is not precisely localized.

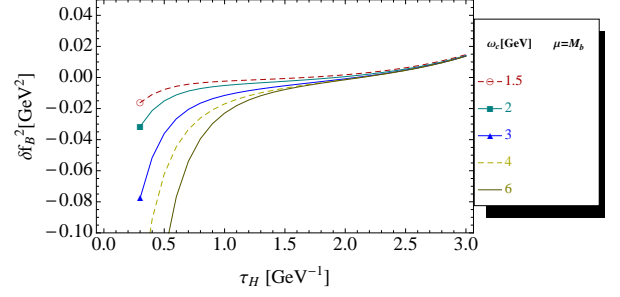


Figure 5: τ_H -behaviour of δf_B^2 for $\mu = M_b$ and for different values of ω_c by including the d=5 condensate in the OPE.

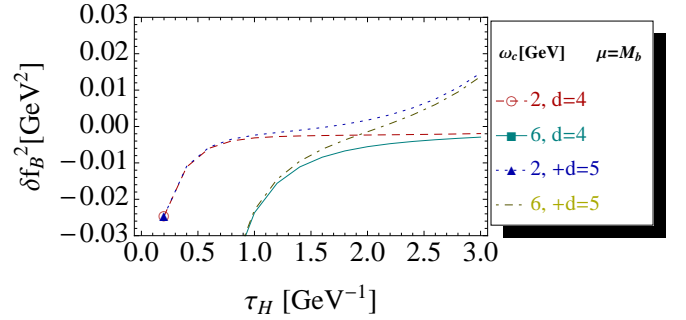


Figure 6: τ_H -behaviour of δf_B^2 for $\mu = M_b$ and $\omega_c = 1.5$ and 6 GeV for different truncations of the OPE by the inclusion of the d=4 condensate or by the inclusion of d=4+5 condensates.

Results for δf_B^2

The “sum rules data” of δf_B^2 for different values of μ are shown Table 3 and in Fig. 7, where the main errors come from the localization of τ_H from 1.8 to 2.2 GeV^{-2} and the one induced from its corresponding ω_c values. The errors from the QCD parameters are negligible. Taking the average of different values,

Table 3: Central values and corresponding errors for δf_B^2 from the LSR at different values of the subtraction point μ and for $M_b = 4804(50)$ MeV.

μ [GeV]	$-\delta f_B^2 \cdot 10^3 [\text{GeV}^2]$	Error $10^3 [\text{GeV}^2]$
1	-0.27	2.79
2	-1.19	3.85
3	-0.66	3.58
4	-0.40	3.45
M_b	-0.27	3.45
5	-0.13	3.32
6	-0.27	3.45

we deduce from the analysis the $1/M_b$ corrections:

$$\delta f_B^2 = -0.48(1.37) \times 10^{-3} \text{ GeV}^2. \quad (38)$$

One should notice that the mass correction from the central value of δf_B^2 only decreases the value of f_B by about 1 MeV which is negligible.

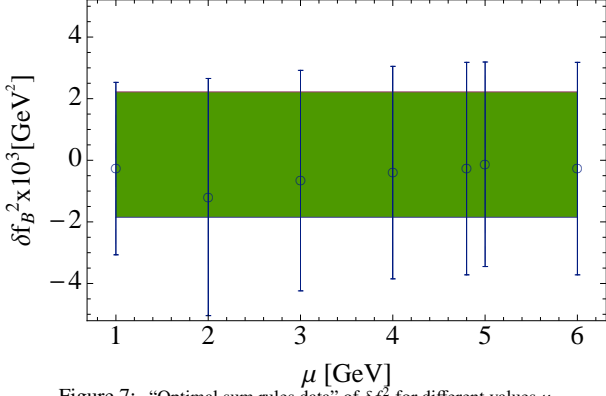


Figure 7: “Optimal sum rules data” of δf_B^2 for different values μ .

7. f_B from HQET and from full QCD

Combining the result in Eq. (38) with the one in Eq. (28) with the help of Eq. (31), one obtains:

$$f_B^{hqet} = 197(13) \text{ MeV}, \quad (39)$$

where we have added the errors quadratically. Notice that, unlike the full QCD case [1], we have not tried to extract an upper bound on f_B from the positivity of the spectral function because of the indefinite sign of δf_B^2 in Eq. (eq:fbphys). We can compare this result with the one obtained in the static limit in Eq. (30), where one can see that the main corrections are to the $(M_b/M_B)^{3/2}$ ratio in Eq. (31). The one due to δf_B^2 only decreases the value of f_B by about 1 MeV. One can also compare the HQET result with the one from the average of LSR and Moment sum rules in full QCD [1]:

$$f_B^{qcd} = 206(7) \text{ MeV}. \quad (40)$$

Taking the average of the two sum rule predictions in Eqs. (39) and (40), we deduce the final averaged prediction from QSSR:

$$f_B^{qssr} = 204(6) \text{ MeV} \quad (41)$$

8. Extraction of the b -quark mass from HQET

One can extract the meson-quark mass-difference Δ using the ratio of the LSR obtained from Eq. (31):

$$\mathcal{R}_{\tau_H} \equiv \frac{-\frac{\partial}{\partial \tau_H} (\hat{f}_B^2 e^{-\tau_H \Delta M})}{\hat{f}_B^2 e^{-\tau_H \Delta M}} = \Delta M \equiv \frac{M_B^2 - M_b^2}{M_b}, \quad (42)$$

where M_b is the on-shell b -quark mass. We notice that the $1/M_b$ corrections encoded in δf_B^2 which contributes about 0.9% of the value of f_B^2 , are also negligible for the extraction of ΔM though act to the τ_H stability analysis of f_B^2 . We show the τ_H -behaviour of ΔM in Fig. 8. We show in Table 4 the different sources of errors on ΔM , where one can notice that the most important ones come from t_c , the estimated α_s^3 and the mixed condensate contributions. We show in Fig. 9 the μ -behaviour of different QSSR data points from which we deduce the average:

$$\Delta M = 853(63) \text{ MeV}. \quad (43)$$

Table 4: Central values and corresponding errors for ΔM in units of MeV from the LSR at different values of the subtraction point μ in units of GeV. The + (resp. -) sign means that the values of ΔM increase (resp. decrease) when the input increases (resp. decreases). The total error comes from a quadratic sum.

μ	ΔM	t_c	α_s	α_s^3	M_b	$\langle \bar{u}u \rangle$	$\langle G^2 \rangle$	M_0^2	Total
1	941	+124	+10	+21	+1	-7	-5	+22	128
2	877	+144	+9	+51	+1	-9	-7	+27	156
3	841	+147	+9	+58	+1	-11	-8	+27	161
4	817	+149	+8	+55	+2	-11	-9	+27	162
5	798	+149	+8	+64	+2	-12	-10	+27	165
6	784	+149	+8	+65	+2	-12	-10	+26	166

Using the previous value of ΔM , one can extract the on-shell b -quark mass to order α_s^2 :

$$M_b^{hqet} = 4871(30) \text{ MeV}. \quad (44)$$

Using the known relation between the on-shell and running quark mass to order α_s^2 (see e.g. [6–8, 32]), we deduce:

$$\bar{m}_b(\bar{m}_b)^{hqet} = 4234(56) \text{ MeV}. \quad (45)$$

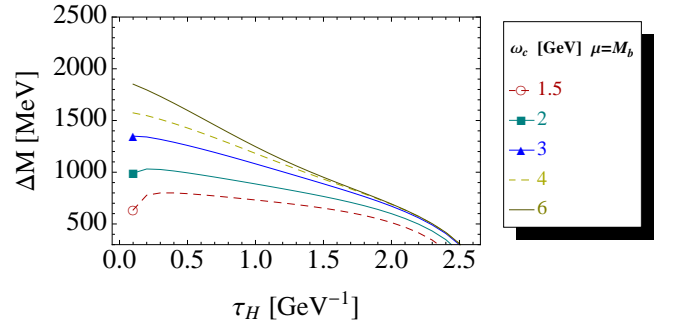


Figure 8: τ_H -behaviour of Δ for $\mu = M_b$ and for different values of ω_c .

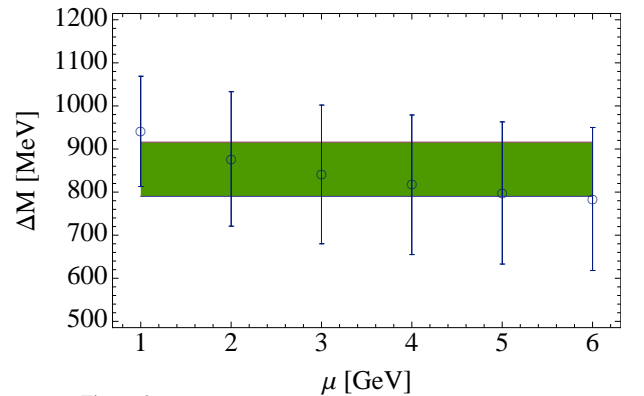


Figure 9: “Optimal sum rules data” of Δ for different values μ .

9. $\bar{m}_b(\bar{m}_b)$ from HQET and from full QCD

We can compare the previous value of the running mass with the one from heavy-light QCD spectral sum rules full QCD to order α_s^2 [1]:

$$\bar{m}_b(\bar{m}_b)^{qcd} = 4236(69) \text{ MeV}, \quad (46)$$

and from the Υ sum rules to order α_s^3 [2]:

$$\bar{m}_b(\bar{m}_b)^{\Upsilon} = 4177(11) \text{ MeV}. \quad (47)$$

We can deduce the *final averaged estimate from QSSR*:

$$\bar{m}_b(\bar{m}_b)^{QSSR} = 4181(11) \text{ MeV}. \quad (48)$$

10. Summary and conclusions

We have re-estimated f_B and M_b from HQET Laplace spectral sum rules to order α_s^2 . Our results in Eqs. (39) and (45) are in good agreement with the ones from full QCD in Eqs. (40) and (46). We have combined different QSSR results including the value of $\bar{m}_b(\bar{m}_b)$ from the Υ systems in Eq. (47) for deducing the *final averaged estimate from QSSR* in Eqs. (41) and (48). These results are in perfect agreement with some other determinations and with lattice results including $n_f = 2$ or 3 dynamical quarks compiled in Table 5 [3, 52–56].

Table 5: Results for f_B and $\bar{m}_b(\bar{m}_b)$ in units of MeV and comparison with lattice simulations using $n_f = 2$ [53, 54] and $n_f = 3$ [55, 56] dynamical quarks. f_P are normalized as $f_\pi = 130.4$ MeV.

Observables	Methods	Refs.
f_B		
<i>QSSR</i>		
$197(13) \equiv 1.58(5)f_\pi$	HQET	(this work)
$206(7) \equiv 1.58(5)f_\pi$	full QCD	[1]
$204(6) \equiv 1.57(5)f_\pi$	Average	(this work)
$\leq 235.3(3.8) \equiv 1.80(3)f_\pi$	full QCD	[1]
<i>Lattice</i>		
197(10)	ETMC	[53]
193(10)	ALPHA	[54]
190(13)	HPQCD	[55]
197(9)	FNAL	[56]
$\bar{m}_b(\bar{m}_b)$		
<i>QSSR</i>		
4234(56)	B -meson - HQET	(this work)
4236(69)	B -meson - full QCD	[1]
4177(11)	Υ - full QCD	[2]
4181(11)	Average	(this work)
<i>Lattice</i>		
4290(140)	ETMC	[53]

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